

Brane-world black holes and energy-momentum vector

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ABSTRACT: The Brane-World black hole models are investigated to evaluate their relative energy and momentum components. We consider Einstein and Møller's energy-momentum prescriptions in general relativity, and also perform the calculation of energy-momentum density in Møller's tetrad theory of gravity. For the Brane-World black holes we show that although Einstein and Møller complexes, in general relativity give different energy relations, they yield the same results for the momentum components. In addition, we also make the calculation of the energy-momentum distribution in teleparallel gravity, and calculate exactly the same energy as that obtained by using Møller's energy-momentum prescription in general relativity. This interesting result supports the viewpoint of Lessner that the Møller energy-momentum complex is a powerful concept for the energy and momentum. We also give five different examples of Brane-World black holes and find the energy distributions associated with them. The result calculated in teleparallel gravity is also independent of the teleparallel dimensionless coupling constant, which means that it is valid in any teleparallel model. **This study also sustains the importance of the energy-momentum definitions in the evaluation of the energy distribution of a given space-time, and supports the hypothesis by Cooperstock that the energy is confined to the region of non-vanishing energy-momentum tensor of matter and all non-gravitational fields.**

KEYWORDS: Classical Theories of Gravity, Black Holes, Black Holes in String Theory.

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1. Introduction

Randall and Sundrum [1, 2] introduced a model that captures some of the essential features of the dimensional reduction of eleven-dimensional super-gravity proposed by Hořava and Witten [3, 4]. The second Randall-Sundrum scenario [2] is a five dimensional anti-de Sitter bulk spacetime with an embedded Minkowski 3-brane where matter fields are confined and Newtonian gravity is effectively re-produced at low energies [5]. The second Randall-Sundrum scenario was generalized to a Friedmann-Robertson-Walker brane, showing that the Friedmann equation at high energies gives $H^2 \sim \rho^2$, in contrast with the general relativistic behavior $H^2 \sim \rho$ [6–8].

The gravitational field on the brane is defined by the modified Einstein equations obtained by Shiromizu, Maeda and Sasaki [9] from five-dimensional gravity with the help of the Gauss and Codazzi equations [10]:

$$G_{\mu\nu} = -\Lambda_4 \delta_{\mu\nu} - \kappa_4^2 T_{\mu\nu} - \kappa_5^4 \Pi_{\mu\nu} - E_{\mu\nu} \tag{1.1}$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} \delta_{\mu\nu} R \tag{1.2}$$

is the four-dimensional Einstein tensor, Λ_4 is the four-dimensional cosmological constant expressed in terms of the five-dimensional cosmological constant Λ_5 and the brane tension ϱ

$$\Lambda_4 = \frac{\kappa_5^2}{2} \left\{ \Lambda_5 + \frac{\varrho^2 \kappa_5^2}{6} \right\}, \tag{1.3}$$

Here, $\kappa_4^2 = 8\pi G_N = \varrho \kappa_5^2 / 6\pi$ is the four-dimensional gravitational constant (here G_N is Newton’s constant of gravity); $T_{\mu\nu}$ is the stress energy tensor of the matter confined on

the brane; $\Pi_{\mu\nu}$ is a tensor quadratic in $T_{\mu\nu}$, obtained from matching the five-dimensional metric across the brane

$$2\Pi_{\mu}^{\nu} = T_{\mu}^{\beta}T_{\beta}^{\nu} - TT_{\mu}^{\nu} - \delta_{\mu}^{\nu} \left(T_{\gamma\rho}T^{\gamma\rho} - \frac{T^2}{2} \right) \quad (1.4)$$

where $T = T_{\mu}^{\mu}$; and $E_{\mu\nu}$ is the electric part of the five-dimensional Weyl tensor projected onto the brane: in proper five-dimensional coordinates, $E_{\mu\nu} = \delta_{\mu}^M \delta_{\nu}^N C_{MNKL} n^K n^L$ where M, N, \dots are five-dimensional indices and n^M is the unit normal to the brane [11]. Our interest here is in selecting a general class of static, spherically symmetric black holes to equation (1.1) without specifying $E_{\mu\nu}$.

The general static, spherically symmetric four-dimensional line-element in the curvature coordinates has the following form,

$$ds^2 = \mathfrak{S}(r)dt^2 - \frac{r}{\Xi(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (1.5)$$

In particular examples we mostly deal with asymptotically flat vacuum solutions, such that $\Lambda_4 = T_{\mu}^{\nu} = 0$. If we treat equation (1.1) as the conventional Einstein equations with an effective stress energy tensor $\hat{T}_{\mu\nu}$ (the hat means effective); i.e.

$$\hat{G}_{\mu\nu} = -\kappa_4^2 \hat{T}_{\mu\nu}, \quad (1.6)$$

then we get

$$\hat{T}_0^0 = \hat{\rho} = \frac{1}{\kappa_4^2 r^3} [1 - r\partial_r \Xi(r)], \quad (1.7)$$

$$\hat{T}_1^1 = \hat{\rho}_{\text{rad}} = \frac{1}{\kappa_4 r^2} \left[\frac{1 - \Xi(r)}{r} + \frac{\Xi(r)\partial_r \mathfrak{S}(r)}{\mathfrak{S}(r)} \right], \quad (1.8)$$

$$\begin{aligned} \hat{T}_2^2 = \hat{T}_3^3 = \hat{\rho}_{\text{tang}} = & \frac{1}{\kappa_4^2} \left[\frac{2\partial_r^2 \mathfrak{S}(r)}{\mathfrak{S}(r)} - \frac{[\partial_r \mathfrak{S}(r)]^2}{\mathfrak{S}(r)} + \frac{\partial_r \mathfrak{S}(r)[r\partial_r \Xi(r) - \Xi(r)]}{r\mathfrak{S}(r)\Xi(r)} \right. \\ & \left. + \frac{2}{r^2} \left(\partial_r \mathfrak{S}(r) - \frac{\partial_r \Xi(r) - \Xi(r)}{r\Xi(r)} \right) \right] \end{aligned} \quad (1.9)$$

Hence, it is very interesting to discuss energy associated with this general brane-world black hole model. In this study to calculate energy in this brane-world black hole solution, we focus on the Møller and Einstein energy-momentum formulations in general relativity and also on Møller's formulation in teleparallel gravity.

Since Einstein introduced general relativity, relativists have not been able to agree upon a definition of the energy-momentum distribution associated with the gravitational field [12]. Einstein [13] first obtained such an expression and many others such as Landau-Lifshitz, Papapetrou, Weinberg, Bergmann-Thomson, Tolman, Møller and Qadir-Sharif gave similar prescriptions [14]. The expressions they gave are called energy-momentum complexes because they can be expressed as a combination of energy-momentum density which is usually defined by a second rank tensor T_i^k and a pseudo-tensor, which is interpreted to represent the energy and momentum of the gravitational field. These formulations have been heavily criticized because they are non-tensorial, i.e. they are coordinate

dependent. Except for the Møller definition these formulations only give meaningful results if the calculations are performed in *Cartesian* coordinates. Møller proposed a new expression for the energy-momentum complex which could be utilized to any coordinate system. Next, Lessner [15] argued that the Møller prescription is a powerful concept for energy-momentum in general relativity.

Recently, the problem of energy-momentum localization has also been considered in teleparallel gravity [16, 17]. Møller showed that a tetrad description of a gravitational field equation allows a more satisfactory treatment of the energy-momentum complex than does general relativity. Therefore, we have also applied the super-potential method by Mikhail *et. al.* [18] to calculate the energy of the central gravitating body. Vargas [16], using the definitions of Einstein and Landau-Lifshitz in teleparallel gravity, found that the total energy is zero in Friedmann-Robertson-Walker space-times. After this work there are a few papers on the energy-momentum in teleparallel gravity [19–21].

Considerable efforts have also been performed in constructing super-energy tensors [22]. Motivated by the works of Bel [23] and independently of Robinson [24], many investigations have been carried out in this field [25].

The paper is organized as follows. In the next section 2, we calculate energy in general relativity by using Einstein and Møller’s energy-momentum formulations. Section 3 gives us the energy of a given model in Møller’s tetrad theory of gravity. Next, in section 4, we give some examples. Finally, last section 5 is devoted to discussions. In this paper we use convention that $G = 1$ and $c = 1$. Except for the cases, we give the special values of the indices, all indices take the values from 0 to 3.

2. Energy-momentum distribution in general relativity

Virbhadra and collaborators revived the interest in this approach [26, 27] and since then numerous works on evaluating the energy and momentum distributions of several gravitational backgrounds have been completed [28–32]. Later attempts to deal with this problematic issue were made by proposers of quasi-local approach. The determination as well as the computation of the quasi-local energy and quasi-local angular momentum of a (2+1)-dimensional gravitational background were first presented by Brown, Creighton and Mann [33]. A large number of attempts since then have been performed to give new definitions of quasi-local energy in Einstein’s theory of general relativity [34]. Furthermore, according to the Cooperstock hypothesis [35], the energy is confined to the region of non-vanishing energy-momentum tensor of matter and all non-gravitational fields.

2.1 Einstein’s 4-momentum formulation

The formulation of the Einstein prescription [13] is defined as

$$\wp_{\mu}^{\nu} = \frac{1}{16\pi} \Delta_{\mu,\alpha}^{\nu\alpha} \tag{2.1}$$

where

$$\Delta_{\mu}^{\nu\alpha} = \frac{g_{\mu\beta}}{\sqrt{-g}} \left[-g(g^{\nu\beta} g^{\alpha\xi} - g^{\alpha\beta} g^{\nu\xi}) \right]_{,\xi} \tag{2.2}$$

φ_0^0 is the energy density, φ_k^0 (where $k=1,2,3$) are the momentum density components, and φ_0^k are the components of energy-current density. The Einstein energy and momentum density satisfies the local conservation laws

$$\frac{\partial \varphi_\mu^\nu}{\partial x^\nu} = 0 \tag{2.3}$$

and the energy-momentum components are given by

$$P_\mu = \int \int \int \varphi_\mu^0 dx dy dz \quad (\mu = 0, 1, 2, 3). \tag{2.4}$$

P_μ is called the momentum four-vector; P_k (here $k=1,2,3$) give momentum components P_1, P_2, P_3 and P_0 gives the energy.

In order to use the Einstein energy-momentum complex, we have to transform the line element (1.5) in quasi-Cartesian coordinates. According to

$$x = r \sin \theta \cos \phi, \tag{2.5}$$

$$y = r \sin \theta \sin \phi, \tag{2.6}$$

$$z = r \cos \theta, \tag{2.7}$$

one gets

$$ds^2 = \mathfrak{S}(r) dt^2 - (dx^2 + dy^2 + dz^2) - \frac{r - \Xi(r)}{r^2 \Xi(r)} (x dx + y dy + z dz)^2. \tag{2.8}$$

Using the metric transformed into quasi-Cartesian coordinates with equations (2.1), (2.2) and (2.4), one gets the following expression for the energy distribution

$$E_{\text{Einstein}}^{GR}(r) = \frac{1}{2} \sqrt{\frac{r \mathfrak{S}}{\Xi}} (r - \Xi), \tag{2.9}$$

and for the momentum components, we have

$$\vec{P}_{\text{Einstein}}^{GR}(r) = 0. \tag{2.10}$$

2.2 Møller's 4-momentum definition

In general relativity, Møller's energy-momentum complex is given by [14]

$$\Upsilon_\mu^\nu = \frac{1}{8\pi} \frac{\partial \Theta_\mu^{\nu\alpha}}{\partial x^\alpha} \tag{2.11}$$

satisfying the local conservation laws:

$$\frac{\partial \Upsilon_\mu^\nu}{\partial x^\nu} = 0 \tag{2.12}$$

where the antisymmetric super-potential $\Theta_\mu^{\nu\alpha}$ is

$$\Theta_\mu^{\nu\alpha} = \sqrt{-g} \left\{ \frac{\partial g_{\mu\beta}}{\partial x^\gamma} - \frac{\partial g_{\mu\gamma}}{\partial x^\beta} \right\} g^{\nu\gamma} g^{\alpha\beta}. \tag{2.13}$$

The locally conserved energy-momentum complex Υ_μ^ν (here $\mu, \nu=0,1,2,3$) contains contributions from the matter, non-gravitational and gravitational fields. Υ_0^0 is the energy density and Υ_i^0 (here $i=1,2,3$) are the momentum density components. The momentum four-vector definition of Møller is given by

$$P_\mu = \int \int \int \Upsilon_\mu^0 dx dy dz \quad (\mu = 0, 1, 2, 3). \quad (2.14)$$

Using Gauss's theorem, this definition transforms into

$$P_\mu = \frac{1}{8\pi} \int \int \Theta_\mu^{0i} \zeta_i dS \quad (2.15)$$

where ζ_i (where $i=1,2,3$) is the outward unit normal vector over the infinitesimal surface element dS . P_i give momentum components P_1, P_2, P_3 and P_0 gives the energy.

Next, using equation (2.13) with the metric (1.5), we obtain the following non-zero component of the antisymmetric super-potential $\Theta_\mu^{\nu\alpha}$

$$\Theta_0^{01}(r, \theta) = \frac{r^2 \mathfrak{S}'(r) \sin \theta}{\sqrt{\frac{r \mathfrak{S}(r)}{\Xi(r)}}}, \quad (2.16)$$

while the momentum density distributions take the form

$$\Upsilon_1^0 = 0, \quad (2.17)$$

$$\Upsilon_2^0 = 0, \quad (2.18)$$

$$\Upsilon_3^0 = 0. \quad (2.19)$$

Hence, the Møller energy distribution is

$$E(r) = \frac{r^{3/2}}{2} \mathfrak{S}'(r) \sqrt{\frac{\Xi(r)}{\mathfrak{S}(r)}}. \quad (2.20)$$

which is also the energy (mass) of the gravitational field that a neutral particle experiences at a finite distance r . Additionally, we can find the momentum components as following

$$P_1^{(Moller)} = P_2^{(Moller)} = P_3^{(Moller)} = 0. \quad (2.21)$$

3. Relative energy in teleparallel gravity

Teleparallel theories, whose basic entities are tetrad fields $h_{a\mu}$ (a and μ are $SO(3,1)$ and space-time indices, respectively) have been considered a long time ago by Møller [36] in connection with attempts to define the energy of the gravitational field. Teleparallel theories of gravity are defined on Weitzenböck space-time [37], which is endowed with the affine connection $\Gamma_{\mu\nu}^\lambda = h^{a\lambda} \partial_\mu h_{a\nu}$ and where the curvature tensor, constructed out of this connection, vanishes identically. This connection defines a space-time with an absolute parallelism or teleparallelism of vector fields [38]. In this geometrical framework the gravitational effects are due to the space-time torsion corresponding to the above mentioned

connection. As remarked by Hehl [39], by considering Einstein's general relativity as the best available alternative theory of gravity, its teleparallel equivalent is the next best one. Therefore it is interesting to perform studies of the spacetime structure as described by the teleparallel gravity.

The super-potential of Møller's in teleparallel gravity is given by Mikhail *et al.* [18] as

$$M_{\mu}^{\nu\beta} = \frac{(-g)^{1/2}}{2\kappa} \left(\delta_{\chi}^{\tau} \left[\delta_{\rho}^{\nu} \delta_{\sigma}^{\beta} - \delta_{\sigma}^{\nu} \delta_{\rho}^{\beta} \right] + \delta_{\rho}^{\tau} \left[\delta_{\sigma}^{\nu} \delta_{\chi}^{\beta} - \delta_{\chi}^{\nu} \delta_{\sigma}^{\beta} \right] - \delta_{\sigma}^{\tau} \left[\delta_{\chi}^{\nu} \delta_{\rho}^{\beta} - \delta_{\rho}^{\nu} \delta_{\chi}^{\beta} \right] \right) \{ \Phi^{\rho} g^{\sigma\chi} g_{\mu\tau} - \lambda g_{\tau\mu} \xi^{\chi\rho\sigma} - (1 - 2\lambda) g_{\tau\mu} \xi^{\sigma\rho\chi} \} \quad (3.1)$$

here $\xi_{\alpha\beta\mu}$ is the con-torsion tensor given by

$$\xi_{\alpha\beta\mu} = h_{i\alpha} h^i{}_{\beta;\mu} \quad (3.2)$$

where the semicolon denotes covariant differentiation with respect to Christoffel symbols

$$\{^{\alpha}_{\mu\nu}\} = \frac{1}{2} g^{\alpha\beta} (\partial_{\mu} g_{\beta\nu} + \partial_{\nu} g_{\beta\mu} - \partial_{\beta} g_{\mu\nu}), \quad (3.3)$$

and Φ_{μ} is the basic vector field defined by

$$\Phi_{\mu} = \xi^{\rho}_{\mu\rho}, \quad (3.4)$$

κ is the Einstein constant, and λ is a free dimensionless parameter.

In a space-time with absolute parallelism the teleparallel vector fields $h_j{}^{\mu}$ define the non-symmetric connection

$$\Gamma^{\alpha}_{\mu\beta} = h_i{}^{\alpha} \partial_{\beta} h^i{}_{\mu} \quad (3.5)$$

The curvature tensor which is defined by $\Gamma^{\alpha}_{\mu\beta}$ is identically vanishing. Møller constructed a gravitational theory based on this space-time. In this gravitation theory the field variables are the 16 tetrad components $h_j{}^{\mu}$, from which the metric tensor is defined by

$$g^{\alpha\beta} = h_j{}^{\alpha} h_j{}^{\beta} \eta^{ij}. \quad (3.6)$$

We assume an imaginary value for the vector $h_0{}^{\mu}$ in order to have a Lorentz signature. We note that, associated with any tetrad field $h_j{}^{\mu}$, there is a metric field defined uniquely by equation (3.6), while a given metric $g^{\alpha\beta}$ doesn't determine the tetrad field completely; for any local Lorentz transformation of the tetrads $h_j{}^{\mu}$ leads to a new set of tetrads which also satisfy equation (3.6).

The energy-momentum density is given by [36]

$$\Xi_{\alpha}^{\beta} = M_{\alpha,\lambda}^{\beta\lambda} \quad (3.7)$$

where comma denotes ordinary differentiation. The energy distribution E and momentum components P_i are expressed by the volume integral [36],

$$\text{Moller } E^{TP} = \lim_{r \rightarrow \infty} \int_{r=\text{constant}} \Xi_0^0 dx dy dz, \quad (3.8)$$

$$\text{Moller } P_i^{TP} = \lim_{r \rightarrow \infty} \int_{r=\text{constant}} \Xi_i^0 dx dy dz. \quad (3.9)$$

Here, the index of i takes the value from 1 to 3 and TP means *Teleparallel Gravity*. The angular momentum J_i of a general relativistic system is given by [36]

$$J_i = \lim_{r \rightarrow \infty} \int_{r=\text{constant}} (x_j \Xi_k^0 - x_k \Xi_j^0) dx dy dz \quad (3.10)$$

where i, j and k take cyclic values 1, 2 and 3. We are interested in determining the total energy, and the momentum components.

In the Cartesian coordinates, the general form of the tetrad h_i^μ which has spherical symmetry, is given [40] as

$$\begin{aligned} h_0^0 &= iG_1, & h_a^0 &= G_2 x^a, & h_a^\alpha &= iG_3 x^\alpha, \\ h_a^\alpha &= G_4 \delta_a^\alpha + G_5 x^a x^\alpha + \epsilon_{\alpha\alpha\beta} G_6 x^\beta, \end{aligned} \quad (3.11)$$

here G_i ($i = 1, 2, 3, 4, 5, 6$) are functions of t and $r = (x^b x^b)^{1/2}$, and the zeroth vector h_0^μ has the factor $i^2 = -1$ to preserve Lorentz signature, and the tetrad of Minkowski space-time is $h_b^\mu = \text{diag}(i, \delta_b^\alpha)$ where ($b=1,2,3$). Using the general coordinate transformation, we write

$$h_{a\mu} = \frac{\partial Y^{\nu'}}{\partial Y^\mu} h_{a\nu'} \quad (3.12)$$

where $\{Y^\mu\}$ and $\{Y^{\nu'}\}$ are, respectively, the isotropic and Schwarzschild coordinates (t, r, θ, ϕ) . In the spherical, static and isotropic coordinate system

$$\begin{aligned} Y^1 &= r \sin \theta \cos \phi, \\ Y^2 &= r \sin \theta \sin \phi, \\ Y^3 &= r \cos \theta. \end{aligned} \quad (3.13)$$

Therefore, we obtain the tetrad components of h^a_μ as

$$h^a_\mu = \begin{pmatrix} \sqrt{\Xi} & 0 & 0 & 0 \\ 0 & \sqrt{\frac{\Xi}{r}} s \theta c \phi & r c \theta c \phi & -r s \theta s \phi \\ 0 & \sqrt{\frac{\Xi}{r}} s \theta s \phi & r c \theta s \phi & r s \theta c \phi \\ 0 & \sqrt{\frac{\Xi}{r}} c \theta & -r s \theta & 0 \end{pmatrix}, \quad (3.14)$$

and the components of inverse matrix h_a^μ

$$h_a^\mu = \begin{pmatrix} \frac{1}{\sqrt{\Xi}} & 0 & 0 & 0 \\ 0 & \sqrt{\frac{\Xi}{r}} s \theta c \phi & \frac{1}{r} c \theta c \phi & -\frac{s \phi}{r s \theta} \\ 0 & \sqrt{\frac{\Xi}{r}} s \theta s \phi & \frac{1}{r} c \theta s \phi & \frac{c \phi}{r s \theta} \\ 0 & \sqrt{\frac{\Xi}{r}} c \theta & -\frac{1}{r} s \theta & 0 \end{pmatrix}. \quad (3.15)$$

Here, we have introduced the following notation: $s\theta = \sin \theta$, $c\theta = \cos \theta$, $s\phi = \sin \phi$ and $c\phi = \cos \phi$. After making the required calculations [41, 42], we obtain the required Møller's super-potential of $M_\mu^{\nu\beta}$ as given below.

$$M_0^{01}(t, r, \theta) = \frac{r^2}{\kappa} \mathfrak{S}'(r) \sin \theta \sqrt{\frac{\Xi(r)}{r \mathfrak{S}(r)}}, \quad (3.16)$$

and the momentum densities are

$$\Xi_1^0 = \Xi_2^0 = \Xi_3^0 = 0. \tag{3.17}$$

Hence, one can easily find that the energy and momentum components are given by the following expressions.

$$E_{(Moller)}^{TP}(r) = \frac{r^2}{2} \mathfrak{S}'(r) \sqrt{\frac{\Xi(r)}{r \mathfrak{S}(r)}}, \tag{3.18}$$

$$P_1^{(Moller)} = P_2^{(Moller)} = P_3^{(Moller)} = 0. \tag{3.19}$$

This is the same energy and momentum expressions as obtained in general relativity by using the Møller energy-momentum complex. It is also independent of the teleparallel dimensionless coupling constant, which means that it is valid not only in teleparallel equivalent of general relativity but also in any teleparallel model.

4. Examples

In this section, we focus on some special black hole models [10] to evaluate energy and momentum distribution (due to matter and fields including gravitation) associated with them.

a) First, we consider the following form of unknown functions $\mathfrak{S}(r)$ and $\Xi(r)$:

$$\mathfrak{S}(r) = 1 - \frac{2m}{r}, \quad \Xi(r) = \frac{(r - 2m)(r - \lambda_0)}{r - \frac{3m}{2}} \tag{4.1}$$

where $m = \text{constant} > 0$ and λ_0 is an integration constant. This solution is the vacuum case $R \equiv 0$. Thus, the line-element takes the following form:

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \frac{\left(1 - \frac{3m}{2r}\right)}{\left(1 - \frac{2m}{r}\right) \left(1 - \frac{\lambda_0}{r}\right)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \tag{4.2}$$

The Schwarzschild metric is restored in the special case $\lambda_0 = 3m/2$. The metric given above was obtained by Casadio, Fabbri and Mazzacurati [43] in search for new brane-world black holes and by Germani and Maartens [44] as a possible external metric of a homogeneous star on the brane. In case $\lambda_0 > 2m$ the metric (4.2) describes a symmetric traversable wormhole [45]. Next, In case $\lambda_0 < 2m$, as in the schwarzschild metric, $\lambda_0 = 2m$ is a simple horizon, and the spacetime structure depends on the sign of $\mathfrak{N} = \lambda_0 - 3m/2$. If $\mathfrak{N} < 0$, the structure is that of a Schwarzschild black hole, but the space-like curvature singularity is located at $\lambda_0 = 3m/2$ instead of $\lambda_0 = 0$. If $\mathfrak{N} > 0$, the solution describes a non-singular black hole with a wormhole throat at $r = \lambda_0$ (inside the horizon, or, more precisely, it is the minimum value of r at which the model bounces. The corresponding global structure is that of a non-extremal Kerr black hole [43].

In this case, we obtain the following energy expressions:

$$E_{(Einstein)}(r) = \frac{1}{2} \left[\frac{m(r - 4\lambda_0) + 2r\lambda_0}{(4r - 6m)^{1/2}(r - \lambda_0)^{1/2}} \right] \quad (4.3)$$

$$E_{(Moller)}(r) = E_{(Moller)}^{TP}(r) = m \left[\frac{r - \lambda_0}{r - \frac{3m}{2}} \right]^{1/2} \quad (4.4)$$

b) Second, we choose

$$\mathfrak{S}(r) = 1 - \frac{h^2}{r^2}, \quad \Xi(r) = r \left(1 - \frac{h^2}{r^2} \right) \left(1 + \frac{\chi - h}{\sqrt{2r^2 - h^2}} \right) \quad (4.5)$$

where $h = \text{constant} > 0$. This form of the black hole model represents a metric with zero Schwarzschild mass. The sphere $r = h$ is a simple horizon if $\chi > 0$ and a double horizon if $\chi = 0$. In case $\chi < 0$, the function $\Omega(r) = r^{-1}\Xi(r)$ has a simple zero at $r = r' > h$ given by $2r'^2 = h^2 + (h - \chi)^2$ which is a symmetric wormhole throat [45]. In case $\chi = 0$, $r = h$ is a double horizon, but a time-like singularity $\Omega \rightarrow \infty$ takes place at $r = h/\sqrt{2}$. Next, in case $0 < \chi < h$, inside the simple horizon, the function $\Omega(r)$ turns to zero at $r = r'$ which is now between h and $h/\sqrt{2}$, and we obtain a Kerr-like regular black hole structure. The value $\chi = h$ leads to the simplest metric, which may be identified as a Reissner-Nordsöm black hole with zero mass and pure imaginary charge. The spacetime causal structure is Schwarzschild, with a horizon at $r = h$ and a singularity at $r = 0$. In case $\chi > 0$, the casual structure is again Schwarzschild but the singularity due to $\Omega(r) \rightarrow \infty$ occurs at $r = h/\sqrt{2}$.

The present example of zero mass black hole shows that, in the brane-world context, a black hole may exist (at least a solution to the gravitational equation on the brane) without matter and without mass, solely as the tidal effect from the bulk gravity.

Energy expressions associated with this case are computed exactly as

$$E_{(Einstein)}(r) = \frac{1}{2} \sqrt{\frac{r}{1 + \frac{\chi - h}{\sqrt{2r^2 - h^2}}}} \left[r - \left(1 - \frac{h^2}{r^2} \right) \left(1 + \frac{\chi - h}{\sqrt{2r^2 - h^2}} \right) \right] \quad (4.6)$$

$$E_{(Moller)}(r) = E_{(Moller)}^{TP}(r) = \frac{h^2}{r^{3/2}} \left[1 + \frac{\chi - h}{\sqrt{2r^2 - h^2}} \right]^{1/2}. \quad (4.7)$$

c) As the third example, lets consider the following form of the functions $\mathfrak{S}(r)$ and $\Xi(r)$:

$$\mathfrak{S}(r) = \left(1 - \frac{2m}{r} \right)^2, \quad \Xi(r) = \frac{1}{2}(r - \lambda_1)(r - \lambda_2) \quad (4.8)$$

where $m = \text{constant} > 0$ and

$$\lambda_2 = \frac{m\lambda_1}{\lambda_1 - m}. \quad (4.9)$$

This is the extremal Reissner-Nordsöm black hole form, and the metric can be written as

$$ds^2 = \left(1 - \frac{2m}{r} \right)^2 dt^2 - \left(1 - \frac{\lambda_1}{r} \right)^{-1} \left(1 - \frac{\lambda_2}{r} \right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (4.10)$$

We obtain a black hole solution in the only case $r_0 = r_1 = 2m$. It is the extremal Reissner-Nordsöm metric and accordingly the effective stress energy tensor is $\widehat{T}_\mu^\nu \propto r^{-4} \text{diag}(1, 1, -1, -1)$. Other values of r_0 lead either to wormholes (the throat is located at $r = r_0$ if $r_0 > 2m$ or at $r = r_1 > 2m$ in case $2m > r_0 > m$), or to a naked singularity located at $r = 2m$ (when $r_0 < m$) as is confirmed by calculating the Kretschmann scalar [45].

When we consider this case to obtain an exact form for the general energy distribution obtained in sections 2 and 3, we find easily

$$E_{(Einstein)}(r) = \frac{(2m - r) [r^2 + \lambda_1 \lambda_2 - r(2 + \lambda_1 + \lambda_2)]}{2[2r(r - \lambda_1)(r - \lambda_2)]^{1/2}} \quad (4.11)$$

$$E_{(Moller)}(r) = E_{(Moller)}^{TP}(r) = m \left[\frac{2}{r} (r - \lambda_1)(r - \lambda_2) \right]^{1/2} \quad (4.12)$$

d) The fourth example can be the following case:

$$\mathfrak{S}(r) = 1 - \frac{r^2}{s^2}, \quad \Xi(r) = \left(1 - \frac{r^2}{s^2} \right) \left\{ r + \frac{Z}{\sqrt{(2s^2 - 3r^2)^3}} \right\} \quad (4.13)$$

where $s = \text{constant} > 0$ and Z is an integration constant such that $Z = 0$ corresponds to integration from $r = \tilde{r} = a\sqrt{2/3}$ to r . The value $r = \tilde{r}$ is the one where $4\mathfrak{S}(r) + \partial_r \mathfrak{S}(r)$ vanishes.

The above three examples described vacuum asymptotically flat black holes solutions. Now, considering the de Sitter form of the function $\mathfrak{S}(r)$, we can write a solution for a vacuum configuration with a cosmological term, so that $R = 4\Lambda_4 = 12/s^2$, in the region $r < a$.

For this example of the Brane-World black holes one can obtain the energy relations in general relativity and teleparallel gravity as

$$E_{(Einstein)}(r) = \frac{1}{2} \sqrt{\frac{r}{r + \frac{Z}{(2s^2 - 3r^2)^{3/2}}}} \left[r - \left(1 - \frac{r^2}{s^2} \right) \left(r + \frac{Z}{(2s^2 - 3r^2)^{3/2}} \right) \right] \quad (4.14)$$

$$E_{(Moller)}(r) = E_{(Moller)}^{TP}(r) = -\frac{r^{5/2}}{s^2} \left[r + \frac{Z}{(2s^2 - 3r^2)^{3/2}} \right]^{1/2} \quad (4.15)$$

e) In the last example, we have

$$\mathfrak{S}(r) = \left(1 - \frac{2m}{r} \right)^{1/f}, \quad \Xi(r) = r \left(1 - \frac{2m}{r} \right)^{\frac{1}{f}-2} \quad (4.16)$$

where $m = \text{constant} > 0$ and $f \in \mathbb{N}$. We here try to give an example for a metric behaving non-analytically at $r = r_h$. The corresponding line-element is

$$ds^2 = \left(1 - \frac{2m}{r} \right)^{1/f} dt^2 - \left(1 - \frac{2m}{r} \right)^{2-\frac{1}{f}} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (4.17)$$

for any positive integer f in the case of simple horizon. There is also another form of the metric which can be given as

$$ds^2 = \left(1 - \frac{2m}{r}\right)^{2/f} dt^2 - \left(1 - \frac{2m}{r}\right)^{-2} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (4.18)$$

with $m > 0$ and this metric $f \in \mathbb{N}$ reveals a double horizon at $r = 2m$.

For this last example of the Brane-World black holes, we get the following expressions for the Einstein and Møller energy distributions

$$E_{(Einstein)}(r) = \frac{(1 - \frac{2m}{r})^{1/f} r^2 - (r - 2m)^2}{4m - 2r} \quad (4.19)$$

$$E_{(Moller)}(r) = E_{(Moller)}^{TP}(r) = \frac{m}{f} \left(1 - \frac{2m}{r}\right)^{\frac{1}{f}-2}. \quad (4.20)$$

5. Discussions

In the present work, in order to compute the energy and momentum distributions (due to matter and fields including gravity) associated with five different Brane-World black holes, we focus on the Einstein and Møller energy-momentum complexes in general relativity and the teleparallel gravity version of the Møller prescription.

We find that momentum components associated with Brane-World black hole models in three of the methods of calculating energy-momentum distribution of the universe are equal to zero; however, for the energy distributions, we obtain the following relations

$$E_{(Einstein)}(r) = \frac{1}{2} \sqrt{\frac{r\mathfrak{S}}{\Xi}} (r - \Xi) \quad (5.1)$$

$$E_{(Moller)}(r) = E_{(Moller)}^{TP}(r) = \frac{r^{3/2}}{2} \mathfrak{S}'(r) \sqrt{\frac{\Xi(r)}{\mathfrak{S}(r)}}. \quad (5.2)$$

Using the above general expressions, we evaluate the energy-momentum distribution of the five different black hole models which are special cases of the general Brane-World ones.

From eqs.(70-71), one can easily see that the results obtained in teleparallel gravity are also independent of the teleparallel dimensionless coupling parameter, which means that it is valid not only in the teleparallel equivalent of general relativity, but also in any teleparallel model.

Furthermore, we show that the Møller energy distribution both in Einstein's theory of general relativity and teleparallel gravity is the same, and this interesting energy result supports the viewpoint of Lessner that the Møller energy-momentum complex is a powerful concept of energy and momentum. Our results also (a) sustains the importance of the energy-momentum definitions in the evaluation of the energy distribution of a given spacetime, and (b) supports the hypothesis by Cooperstock that the energy is confined to the region of non-vanishing energy-momentum tensor of matter and all non-gravitational fields.

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